Theorem: $V_h = \{ u_h \in H^1(\Omega),\ u_h|_K \in \mathbb{P}_1(K) \}$
$H^1_0(\Omega)$

Then $V_h = \text{span}\{ \varphi^v,\ v \text{ internal vertices } \mathcal{T}_h \}$

$\varphi^v = \begin{cases} 1 & \text{at } v \\ 0 & \text{at all other vertices.} \end{cases}$

proof:

·) $V_h \supseteq \text{span}\{-\}$, $\varphi^v \in V_h$ (trivial).

·) $V_h \subseteq \text{span}\{-\}$

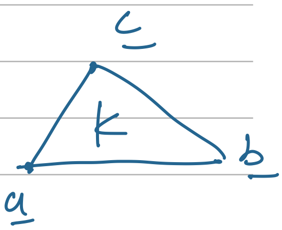
Let $u \in V_h$, $u(v_i) = c_i$, v_i a vertex of $\mathcal{T}_h$.

Consider $w(x) = \sum_i c_i \varphi^{v_i}(x) \in \text{span}\{-\}$

$$(u-w)(v_j) = u(v_j) - \sum_i c_i \underbrace{\varphi^{v_i}(v_j)}_{=\delta_{ij}}$$

$$= u(v_j) - c_j = 0$$

$\forall K \in \mathcal{T}_h$ $\quad (u-w)|_K(v_i) = 0$



$$(u-w)(\underline{a}) = (u-w)(\underline{b})$$

$$= (u-w)(\underline{c}) = 0$$

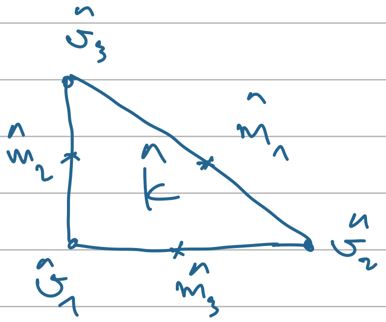
and $(u-w)|_K \in \mathbb{P}_1(K)$

$$\Rightarrow (u-w)|_K = 0 \quad \forall K \in \mathcal{T}_h$$

$$\Rightarrow u(x) = w(x) = \sum_i c_i \varphi^{v_i}(x).$$ ∎

Construction of $\mathbb{P}_2$ FEM in 2D:

$\dim \mathbb{P}_2(\mathbb{R}^2) = 6$



$$\begin{cases} \hat{\varphi}^{\hat{v}_i}(x) \in \mathbb{P}_2(\hat{K}) \\ \hat{\varphi}^{\hat{v}_i}(\hat{v}_j) = \delta_{ij}, \quad i,j = 1,2,3 \end{cases}$$

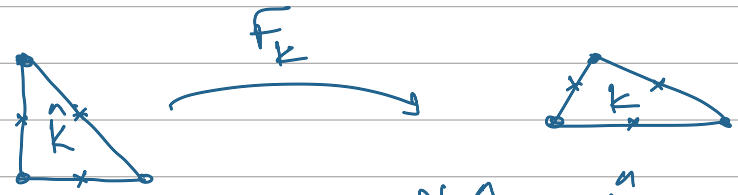
$$\begin{cases} \hat{\varphi}^{\hat{m}_j}(\hat{m}_j) \in \mathbb{P}_2(\hat{K}) \\ \hat{\varphi}^{\hat{m}_i}(\hat{m}_j) = \delta_{ij}, \quad i,j = 1,2,3 \end{cases}$$

and
$$\begin{cases} \hat{\varphi}^{m_i}(\hat{v}_j) = 0, \quad j = 1,2,3 \\ \hat{\varphi}^{\hat{v}_i}(\hat{m}_\ell) = 0, \quad \ell = 1,2,3 \end{cases}$$

$$\mathbb{P}_2(\hat{K}) = \operatorname{span}\{\hat{\varphi}^{\hat{v}_i}, \hat{\varphi}^{\hat{m}_\ell} ; i = 1,2,3, \ \ell = 1,2,3\}$$

(the construction is an exercise)

$\Rightarrow$ Affine map to the physical element K.

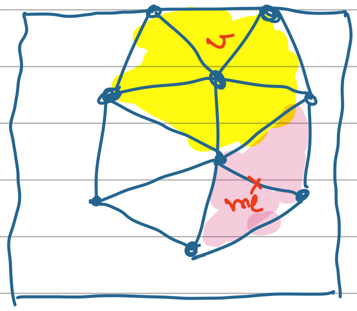


$\forall \hat{x} \in \hat{K} \qquad F_K(\hat{x}) = x$

$$\begin{cases} \varphi^{v_i}(F_K(\hat{x})) = \hat{\varphi}^{\hat{v}_i}(\hat{x}) \\ \varphi^{m_\ell}(F_K(\hat{x})) = \hat{\varphi}^{\hat{m}_\ell}(\hat{x}) \end{cases}$$

$$\Longrightarrow \quad \mathbb{P}_2(K) = \operatorname{span}\{\varphi^{v_i}, \varphi^{m_\ell} ; \ i = 1,2,3, \ \ell = 1,2,3\}$$

Basis for $V_h = \{ u \in H^1(\Omega),\ u|_K \in \mathbb{P}_2(K) \}$



v is a vertex of $\mathcal{T}_h$

- φ^v = the $\mathbb{P}_2$ basis function obtained by collecting all the functions attached to v.

- φ^e = the $\mathbb{P}_2$ basis function obtained by the two functions attached to m_e.

$$\operatorname{supp}(\varphi^v) = \bigcup_{K \in \mathcal{T}_v} K \quad ; \quad \operatorname{supp}(\varphi^{m_e}) = \bigcup_{e \in \mathcal{T}_{m_e}} K$$

<u>Proposition</u>: $V_h = \operatorname{span}\{ \varphi^v, \varphi^e \quad \forall v \text{ vertex of } \mathcal{T}_h,\ \forall e \text{ edge of } \mathcal{T}_h \}$.

<u>proof</u>: Same as for $\mathbb{P}_1$.

<u>Definition</u> The dofs are:

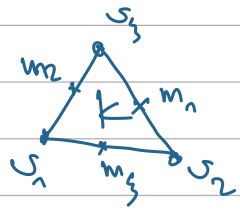
$$\begin{cases} \delta^v(f) = f(v) & \forall v \text{ vertex of } \mathcal{T}_h \\ \delta^e(f) = f(m_e) & \forall e \text{ edge of } \mathcal{T}_h. \end{cases}$$

m_e: mid point of e.

Proposition: This set of dofs is unisolvent on V_h.

proof: $u \in V_h$ $\quad \delta^v(u) = \delta^e(u) = 0$

$\Longrightarrow u = 0$?



$u|_K \in \mathbb{P}_2(K)$ $\quad \begin{cases} (u|_K)(v_i) = 0 \\ (u|_K)(m_e) = 0 \end{cases}$

$\forall i, e = 1, 2, 3$

$\Longrightarrow u|_K = 0 \quad \forall K$

$\Longrightarrow u = 0.$ ■

Definition $f \in C^0(\overline{\Omega})$ we define

$$\Pi_2 f = \sum_{\substack{v \\ \text{vertex} \\ \text{of } \mathcal{T}_h}} f(v)\, \varphi^v(x) + \sum_{\substack{e \\ \text{edge of } \mathcal{T}_h}} f(m_e)\, \varphi^e(x)$$

This operator verifies:

$$\begin{cases} (f - \Pi_2 f)(v) = 0 & \forall v \text{ vertex} \\ (f - \Pi_2 f)(m_e) = 0 & \forall e \text{ edge.} \end{cases}$$

mid points.

Assembly of the FE problem

$$\begin{cases} -\mathrm{div}\,(\underline{\underline{A}}(x)\nabla u) = f & \text{in } \Omega ; \\ u = 0 & \text{on } \partial\Omega . \end{cases} \qquad \begin{cases} A \in L^\infty(\Omega)^{2\times 2} \\ A \text{ is SPD.} \end{cases}$$

$\gamma(u) = 0$

Find $u \in H_0^1(\Omega)$ s.t $\forall v \in H_0^1(\Omega)$

$$\int_\Omega A(x)\nabla u \cdot \nabla v = \int_\Omega f v$$

$$\boxed{V_{h,0}^{(1)} = V_h^{(1)} \cap H_0^1(\Omega)} \quad ; \quad V_{h,0}^{(2)} = V_h^{(2)} \cap H_0^1(\Omega)$$

$\mathbb{P}_1$ F.E $\qquad \mathbb{P}_2$ F.E

$V_{h,0}^{(1)} = \mathrm{span}\{ \varphi^v$ basis of $V_h^{(1)}$, v are internal vertices. $\}$

$$V_h = \mathrm{span}\{\varphi_1, \ldots, \varphi_N\}.$$

$u \in V_{h,0}^{(1)}$, $\quad u(x) = \sum_{i=1}^N c_i \varphi_i$

$$\sum_{i=1}^N c_i \underbrace{\int_\Omega A(x)\nabla\varphi_i \cdot \nabla\varphi_j}_{A_{ij}} = \underbrace{\int_\Omega f \varphi_j}_{f_j} \qquad \forall \varphi_j \in V_h$$

$$\underline{c} = (c_1, c_2, \ldots, c_N)^T \; ; \; \underline{F} = (f_1, \ldots, f_N)^T$$

$$(A)_{ij} = A_{ij}$$

$\Rightarrow$ $A \underline{c} = \underline{F}$ We must solve the linear system

(stiffness matrix) to find $\underline{c}$.

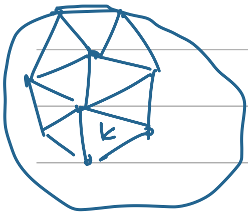
1- strategy to assemble $\underline{A}$?

2- structure of $\underline{A}$?

3- properties of A, conditioning ?

$\rightarrow$ Assembly of A in $\mathbb{P}_1$ F.E

1) $$A_{ij} = \int_\Omega A(x) \nabla\varphi_i \cdot \nabla\varphi_j \, dx$$



$$= \sum_K \int_K A(x) \nabla\varphi_i \cdot \nabla\varphi_j \, dx.$$

$$\begin{cases} (A_K)_{ij} = \int_K A(x) \nabla\varphi_i|_K \cdot \nabla\varphi_j|_K \, dx. \\ (f_K)_j = \int_K f \varphi_j|_K \end{cases}$$

3x3 matrix (local)

2) local to global numbering :

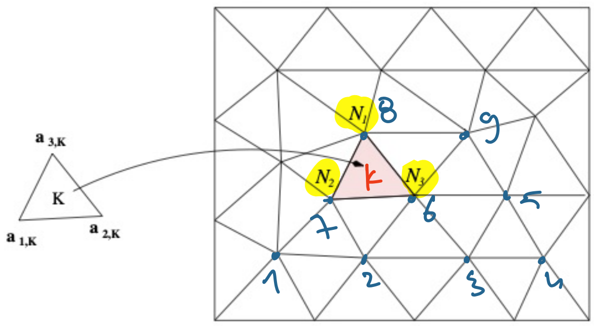


Figure 4.1: Local to global map

local $N_1 \longrightarrow 8$

local $N_2 \longrightarrow 7$

local $N_3 \longrightarrow 6$

$\leftrightarrow$ (connectivity matrix)

triangle K	local numbering		global numbering
	1 (vertex $a_{1,K}$)	$\longrightarrow$	$\mathcal{N}_1$
	2 (vertex $a_{2,K}$)	$\longrightarrow$	$\mathcal{N}_2$
	3 (vertex $a_{3,K}$)	$\longrightarrow$	$\mathcal{N}_3$

$\hookrightarrow$ Algo $\quad$ For all $K \in \mathcal{T}_h$:

$\qquad$ For $i = 1,2,3;\ j = 1,2,3$

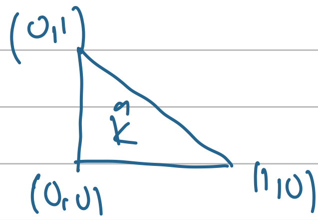
$\qquad\quad$ compute A_K ;

$$A_{N_i N_j} \leftarrow A_{N_i N_j} + (A_K)_{ij} ;$$

$$f_{N_j} \leftarrow f_{N_j} + (f_K)_j ;$$

• How to compute A_K ?

Recall $\quad X = F_K(\hat{X}) \ ; \quad F_K : \hat{K} \longrightarrow K$



s.t $\quad F_K(\hat{X}) = B_K \hat{X} + \underline{b_K}$

$$\hat{X} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}.$$

$$\mathbb{P}_1(\hat{K}) = \operatorname{span}\{\hat{x}, \hat{y}, 1-\hat{x}-\hat{y}\}$$

$$\varphi_i(x) = \varphi_i(F_K(\hat{x})) = \hat{\varphi}_i(\hat{x})$$

$$\nabla\varphi_i(x) = \nabla\varphi_i(F_K(\hat{x})) = B_K^{-T}\,\hat{\nabla}\hat{\varphi}_i(\hat{x})$$

$$dx = |\det B_K|\, d\hat{x}$$

$$(A_K)_{ij} = \int_K A(x)\,\nabla\varphi_i \cdot \nabla\varphi_j \, dx \ , \quad i,j = 1,2,3$$

$$= \int_{\hat{K}} \Big[A(F_K(\hat{x}))\, \underbrace{B_K^{-T}\,\hat{\nabla}}_{\text{a vector.}}\hat{\varphi}_i(\hat{x})\Big] \cdot \Big[B_K^{-T}\,\underbrace{\hat{\nabla}}_{\text{a vector.}}\hat{\varphi}_j(\hat{x})\Big] |\det B_K|\, d\hat{x}$$

$\Lsh$ computed using a quadrature formula on $\hat{K}$.

($\hookrightarrow$ in more details later) !!

Same for $(\mathcal{F}_K)_j = \int_K f(x)\, \varphi_j(x)\, dx.$

$$= \int_{\hat{K}} f(F_K(\hat{x}))\, \hat{\varphi}_j(\hat{x})\, |\det B_K|\, d\hat{x}.$$

• Structure of A :

First, some inequalities :

Euclidean norm.

$$\|v\|^2_{L^2(K)} \simeq h_K^2\, |\underline{\hat{v}}|_2^2 \simeq h_K^2\, \|\hat{v}\|^2_{L^2(\hat{K})}$$

$\simeq$ means $\exists\, c_1, c_2 > 0$ $\left\{ \begin{array}{l} \|v\|^2_{L^2(K)} \leq c_1 h_K^2 |\underline{v}|_2^2 \\ |\underline{\hat{v}}|_2^2 \leq c_2 \|v\|_{L^2(K)} \end{array} \right.$

where :

$$\hat{v}(\hat{x}) = \sum_{\ell=1}^{N_{\hat{K}}} \hat{v}_\ell\, \underline{\hat{\varphi}_\ell}(\hat{x}) \; ; \; \underline{\hat{v}} = [\hat{v}_1, \ldots, \hat{v}_{N_{\hat{K}}}]^T$$

$= v(F_K(\hat{x}))\big|_K$ the restriction of $v \in V_h$ in $K \in \mathcal{T}_h$.

$$\|v\|^2_{L^2(K)} = \int_K |v|^2\, dx \; ; \; \|\hat{v}\|^2_{L^2(\hat{K})} = \int_{\hat{K}} |\hat{v}|^2\, d\hat{x}$$

⇒ Local inverse estimate

$$\exists\, C > 0 \quad \|\nabla v\|^2_{L^2(K)} \leq C \frac{1}{\rho_K^2} \|v\|^2_{L^2(K)} \quad \forall v \in V_h$$

→ in a regular mesh : $\|\nabla v\|_{L^2(K)} \leq C' h_K^{-1} \|v\|_{L^2(K)}$
$(\rho_K \simeq h_K)$